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Supplement to
HIERARCHICAL BASES FOR ELLIPTIC PROBLEMS

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1. Two estimates for nonconforming finite element spaces. The purpose of this section is to prove two estimates for a nonconforming space V in two space dimensions. Concerning V and the triangulation $\mathcal{T}$, we make the following assumptions:

— $\mathcal{T}$ is a regular triangulation of Ω;

— for any $T \in \mathcal{T}$ we have $V|_T \subset C^0(T) \cap H^{1,2}(T)$. For any edge K of the triangulation, denote by v_+, v_- the limits of v on K from different sides and assume that

$$\int_K (v_+ - v_-) = 0$$

(or alternatively, assume that there exists at least one point $p \in K$ such that $v_+(p) = v_-(p)$).

a. A Poincaré estimate. Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary and finitely many corner points $p_1, \ldots, p_m$. Define the distance from the points by

$$d_c(x) := \min\{\text{dist}(x, p_i) : i = 1, \ldots, m\}.$$

Assume that the Dirichlet problem

$$\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

admits an a priori estimate

$$\| u \|_{2,p,0} \leq C_0 \| f \|_{0,p}$$

for $p > 1$, or

$$\| u \|_{2,2,0(\sigma)} \leq C_0 \| f \|_{0,2(\sigma)},$$

where

$$\| u \|_{2,2,0(\sigma)} = \| \nabla^2 u \|_{0,2(\sigma)} := \| d_c^\sigma \nabla^2 u \|_{0,2}$$

and $\sigma < 1$. Denote by P_Ω the Poincaré constant of the domain Ω.

Let Ω_h be a regularly triangulated domain contained in Ω. Let V be a nonconforming finite element space satisfying the assumptions above. Note that

$$\| v \|_V^2 := \sum_{T \in \mathcal{T}} \| v \|_{1,2,0;T}^2$$

defines a norm on V.

LEMMA 1

Under the previous assumptions a Poincaré estimate on V holds More precisely, there is a constant c not depending on $h := \max_{T \in \mathcal{T}} (d_T)$ such that for all $v \in V$

$$\| v \|_0 \leq (P_\Omega + ch^\gamma) \| v \|_V$$

holds, where

$$\gamma = 1 - \frac{2-p}{p} \quad or \quad \gamma = 1 - \sigma.$$

Proof. Let v be an arbitrary element of V and u the solution of

$$\Delta u = v \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Now compute

$$\int_\Omega |v|^2 = \int_\Omega v \Delta u = -\sum_{T \in \mathcal{T}} \int_T \nabla v \cdot \nabla u + \sum_{K \in \mathcal{K}} \int_K v \partial_n u.$$

Utilizing

$$\int_K (v_+ - v_-) = 0,$$

we obtain

$$\sum_{K \in \mathcal{K}} \int_K v \partial_n u = \sum_{K \in \mathcal{K}} \int_K (v - \bar{v})(\partial_n u - \overline{\partial_n u}) \leq \sum_{K \in \mathcal{K}} \| v - \bar{v} \|_{0,q,K} \| \partial_n u - \overline{\partial_n u} \|_{0,p;K} .$$

Here the bar indicates the mean value on K, and p, q are conjugate Hölder exponents. Application of the Sobolev inequalities yields

$$\| v - \bar{v} \|_{0,q,K} \leq C|K|^{\frac{1}{q}+\frac{1}{2}} \| \nabla v \|_{0,2;K} \leq C|K|^{\frac{1}{q}} \| \nabla v \|_{0,2,T} .$$

Here we have also used that $V|_T$ is a finite-dimensional space, and that for every triangle T and any edge K of T we have $|K|^2 \sim |T|$. In the same way we get

$$\| \partial_n u - \overline{\partial_n u} \|_{0,p,K} \leq C|K|^{1-\frac{1}{p}} \| \nabla^2 u \|_{0,p,T}$$

or alternatively

$$\| \partial_n u - \overline{\partial_n u} \|_{0,p,K} \leq C|K|^{\frac{1}{p}-\sigma} \| d_c^\sigma \nabla^2 u \|_{0,2,T} .$$

This gives

$$| \sum_{K \in \mathcal{K}} \int_K v \partial_n u | \leq C \max_{T \in \mathcal{T}} (d_T^\gamma) \| v \|_V \| v \|_0,$$

where γ is as stated above. Observe that

$$\sum_{T \in \mathcal{T}} \int_T \nabla v \cdot \nabla u \leq \| v \|_V \| u \|_{1,0} \leq P_\Omega \| v \|_V \| v \|_0 .$$

Thus,

$$\| v \|_0^2 \leq (P_\Omega + C \max_{T \in \mathcal{T}} (d_T^\gamma)) \| v \|_V \| v \|_0,$$

which yields the required result. *qed*

Remark. In a similar way we can prove a Poincaré inequality for v of zero mean value We derive

$$\| v \|_0 \leq (P_{\Omega,N} + ch^{\tilde{\gamma}}) \| v \|_V$$

for $P_{\Omega,N}$ being the corresponding constant in the case of Sobolev functions. To prove this, consider the solution of the problem: $\Delta u = v$ in Ω, $\partial_n u = 0$ on $\partial\Omega$ and $\frac{1}{|\Omega|} \int_\Omega u = 0$

b. An inverse estimate. Consider first the case of conforming elements. The estimate below is proved in [9] for $n = 2$. The generalization to the case $n > 2$ is obvious, but as the proof for the nonconforming case follows the lines of the present proof, it will be carried out here.

LEMMA 2.

Let T be a regularly refined simplex. Let D be any subsimplex of T and $u \in H^{1,2}(T)$. Then

$$\frac{1}{|D|} \int_D |u| \leq C d_T^{1-\frac{n}{2}} \kappa(d_T / d_D)^{\frac{1}{2}} \| u \|_{1,2,*,T}$$

with κ given by

$$\kappa(\lambda) = \begin{cases} 1 + \log(\lambda), & n = 2, \\ \lambda^{n-2}, n > 2. \end{cases}$$

Proof. We first prove the following:

Let $B_r := \{x \in \mathbb{R}^2 : |x| \le r\}$, $u \in H_0^{1,2}(B_R)$ and $0 < \sigma < \frac{1}{2}R$. Then

$$\frac{1}{|B_\sigma|}\int_{B_\sigma} |u| \;\le\; CR^{1-\frac{n}{2}}\kappa(R/\sigma)^{\frac{1}{2}} \;\| u \|_{1,2,0,B_R} .$$

Introduce polar coordinates: $x \mapsto r\omega \in \mathbb{R}_+ \times S^{n-1}$. Let $f \in L^2(\mathbb{R}_+)$. Consider any $u \in C_0^\infty(B_R)$ and let $v := |u|$. Then

$$\int_{B_R} f(r)v(x)dx = \int_0^R f(r)r^{n-1}\int_{S^{n-1}} v(r\omega)d\omega\, dr.$$

Introducing

$$F(r) = \int_0^r f(t)t^{n-1}dt,$$

we proceed with

$$\int_{B_R} f(r)v(x)dx \;=\; -\int_0^R F(r)\int_{S^{n-1}} \omega\cdot\nabla v(r\omega)d\omega\, dr,$$

after having performed an integration by parts. Using the Cauchy-Schwarz inequality, we get

$$\int_{B_R} f(r)v(x)dx \;\le\; C(\int_0^R \frac{F(r)^2}{r^{n-1}}dr)^{\frac{1}{2}} \;\| v \|_{1,2,0,B_R},$$

where C depends only on n. Letting

$$f(r) \;=\; \begin{cases} \frac{1}{|B_\sigma|}, 0 \le r \le \sigma, \\ 0, \sigma < r, \end{cases}$$

and computing the integral above, meets the requirement.

Return now to the simplices T and D. For $u \in H^{1,2}(T)$ we construct an extension $\hat u \in H_0^{1,2}(B_R)$ of u such that the estimate

$$\| \hat u \|_{1,2,0;B_R} \le C \| u \|_{1,2,*,T}$$

holds. (To do this, we construct a triangulation of $\mathbb{R}^2$ by translating and reflecting T. Extend u as a periodic function. Let $\zeta := \max(0, \min(1 - \mathrm{dist}(\cdot, T), 1))$ and define $\hat u := \zeta u$). Applying the foregoing result to $\hat u$, we obtain the desired result. *qed*

Consider now the situation of a nonconforming finite element space in two dimensions.

LEMMA 3.

For all $u \in V$, and T, D as in the previous lemma, the estimate

$$\frac{1}{|D|}\int_D |u| \;\le\; C(1 + \log(d_T/d_D))^{\frac{1}{2}} \;\| u \|_{V,*,T}$$

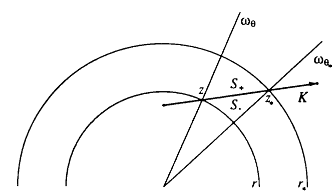
holds.

Proof. Given D, let σ be the radius of the inner circle. For simplicity let 0 be its barycenter. Examining the proof of the previous lemma, we find that we can proceed in nearly the same way. Consider a regular triangulation of a domain containing B_R, and let v be a function in the corresponding finite element space (satisfying the properties above) with $v = 0$ outside B_R. The crucial step where we cannot apply the foregoing proof is the integration by parts. More precisely, if

$$G(r) := \int_0^{2\pi} v(r\omega_\phi)d\phi,$$

then ∂_r does not commute with the integral sign.

To compute this derivative, we consider the following situation:



Let K be an edge of a triangle belonging to the partition of T. Let $z = r\omega_\theta \in B_r \cap K$ We may assume that z is the only point in $B_r \cap K$ in some neighbourhood of the angle θ. For simplicity, let this interval be denoted by $[0, \gamma]$. Let $z_* = r_*\omega_{\theta_*}$ be the point of intersection between B_{r_*} and K. Without loss of generality let $\theta_* < \theta$. We split

$$\int_0^\gamma (v(r_*\omega_\phi) - v(r\omega_\phi))d\phi \;=\; \int_{[0,\gamma]\setminus[\theta_*,\theta]} \cdots + \int_{[\theta_*,\theta]} \cdots .$$

Take $\xi_\phi\omega_\phi$ to be the intersection of the line $\phi = \text{const}$ with K. We know that v is not continuous on this set, and therefore we let

$$v_+(\xi_\phi\omega_\phi) = \lim_{r\to\xi_\phi+0} v(r\omega_\phi),$$

$$v_-(\xi_\phi\omega_\phi) = \lim_{r\to\xi_\phi-0} v(r\omega_\phi).$$

Using this definition, we obtain

$$\int_{\theta_*}^{\theta}(v(r_*\omega_\phi) - v(r\omega_\phi))d\phi = \int_{\theta_*}^{\theta}\{\int_{\xi_\phi}^{r_*}\omega_\phi \cdot \nabla v(\xi\omega_\phi)d\xi + \int_{r}^{\xi_\phi}\omega_\phi \cdot \nabla v(\xi\omega_\phi)d\xi\ \}\ d\phi + \int_{\theta_*}^{\theta}(v_-(\xi_\phi\omega_\phi) - v_+(\xi_\phi\omega_\phi))d\phi.$$

Here, θ_* is a smooth function of r_* and $\lim_{r_*\to r}\theta_* \to \theta$. One verifies readily that for $r_* \to r$

$$\theta_* = \theta + \frac{1}{r}(r_* - r)\nu + O(|r_* - r|^2),$$

where $\nu = -\sqrt{1-(\omega_\theta \cdot e_K)^2}$ and e_K is the unit vector in the direction of the edge K. Note that $r > \sigma$ for any K. Estimating for example the first integral on the right-hand side above, we obtain

$$\int_{\theta_*}^{\theta}\int_{\xi_\phi}^{r_*}\omega_\phi \cdot \nabla v(\xi\omega_\phi)d\xi d\phi \ \le\ \{\int_{\theta_*}^{\theta}\int_{\xi_\phi}^{r_*}\frac{1}{\xi}d\xi d\phi\}^{\frac{1}{2}}\ \| v \|_{1,2,0,S_+}\ .$$

Therefore, integrals of this type are of order $o(|r_* - r|)$ for $r_* \to r$. In the limit $r_* \to r$, we obtain

$$\frac{1}{|r_* - r|}\int_0^{\gamma}(v(r_*\omega_\phi) - v(r\omega_\phi))d\phi \ =\ \int_0^{\gamma}(\omega_\phi \cdot \nabla v(r\omega_\phi))d\phi + \frac{\nu}{r}(v_-(r\omega_\theta) - v_+(r\omega_\theta)).$$

Let S_r denote the set of all points on $B_r \cap K$, $K \subset \mathcal{K}$ (the set of all edges). Then we will find that

$$\partial_r G(r) = \int_0^{2\pi}\omega_\phi \cdot \nabla v(r\omega_\phi)\,d\phi + \frac{1}{r}\sum_{z\in S_r}(v_+(z) - v_-(z))\,\nu_z.$$

Now we only have to discuss the second part of the right-hand side, because with the first part we can proceed as before. It remains to estimate

$$I := \int_0^R \frac{F(r)}{r}\sum_{z\in S_r}(v_+(z) - v_-(z))\,\nu_z dr \ =\ \sum_{K\in\mathcal{K}}\int_K \frac{F(r)}{r}(v_-(z) - v_+(z))\,\nu_z dz,$$

where $F(r) = (\frac{r}{\sigma})^2\chi_{\{r<\sigma\}} + \chi_{\{r>\sigma\}}$. But $r > \sigma$ for any $K \in \mathcal{K}$, and therefore $F(r) = 1$. Let $\bar{v}_\pm$ be the mean value of $v_\pm$ over K. Then we can estimate

$$\begin{aligned}\int_K |v_\pm - \bar{v}_\pm| &\le |K|^{\frac{1}{2}}\{\int_K |v_\pm - \bar{v}_\pm|^2\}^{\frac{1}{2}}\\ &\le C|K|^{\frac{3}{2}}\{\int_K |\nabla v_\pm|^2\}^{\frac{1}{2}}\\ &\le C|K|\{\int_{T_\pm}|\nabla v_\pm|^2\}^{\frac{1}{2}},\end{aligned}$$

where $T_\pm$ are the neighbouring triangles to K. Here we used that v varies in a finite-dimensional space, and that therefore

$$d_T\int_K |\nabla v_\pm|^2 \ \le\ C\int_T |\nabla v_\pm|^2.$$

Notice that the regularity of the triangulation implies $|K|^2 \sim |T|$. While summing over all $T \in \mathcal{T}$, each triangle will be counted three times at most. Therefore, we can estimate I as follows:

$$\begin{aligned}|I| &\le C(\{\int_D |\nabla v|^2\}^{\frac{1}{2}} + \sum_{T\in\mathcal{T}\setminus D}\frac{|T|^{\frac{1}{2}}}{r_T}\{\int_T |\nabla v|^2\}^{\frac{1}{2}})\\ &\le C\,\{1 + \sum_{T\in\mathcal{T}\setminus D}\frac{|T|}{r_T^2}\}^{\frac{1}{2}}\ \| v \|_V,\end{aligned}$$

where r_T is defined as $\min\{r > 0 : r\omega \in T\}$. The sum is actually a Riemann approximation of the integral

$$\int_{B_R\setminus B_\sigma}\frac{dx}{|x|^2} = 2\pi\log(R/\sigma).$$

Finally, we end up with

$$\frac{1}{|D|}\int_D |v| \ \le\ C(1 + \log(R/\sigma))^{\frac{1}{2}}\ \| v \|_V,$$

because we can estimate the mean value of v over B_σ by that over D. Now continue as in the proof before. *qed*